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Let h be the largest altitude of a triangle with circumradius R and inradius r.
Show that $R+r \leq h$.

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As can be seen from the further, the inequality of the problem isn't holds in any triangle but holds in any non-obtuce triangle.

Let $h = h_c = \max\{h_a, h_b, h_c\}$ that is $h_a, h_b \leq h_c \Leftrightarrow \frac{1}{a}, \frac{1}{b} \leq \frac{1}{c} \Leftrightarrow C \leq A, B$.

We have $R+r \leq h = R+r \leq \frac{ab}{2R} \Leftrightarrow 1+\frac{r}{R} \leq \frac{ab}{2R^2} \Leftrightarrow$
$\cos A+\cos B+\cos C \leq 2\sin A\sin B \Leftrightarrow \cos A+\cos B+\cos C \leq \cos(A-B)-\cos(A+B) \Leftrightarrow$
$\cos A+\cos B+\cos C \leq \cos(A-B)+\cos C \Leftrightarrow \cos A+\cos B \leq \cos(A-B)$.

If $C = B = 30^\circ$ and $A = 120^\circ$ then $\cos A+\cos B = \frac{\sqrt{3}-1}{2} > \cos(A-B) = 0$.

Thus, for further we assume that the triangle isn't obtuse and also WLOG assume that $B \leq A$. Then we obtain $\begin{cases} 0 < C \leq B \leq A \leq \pi/2 \\ A+B+C = \pi \end{cases} \Leftrightarrow$

(**1**) $$\begin{cases} \frac{\pi}{3} \leq A \leq \pi/2 \\ \frac{\pi-A}{2} \leq B \leq A \\ C = \pi-A-B \end{cases}.$$

Since $\cos(A-B)$ and $-\cos B$ both increase by $B \in \left[\frac{\pi-A}{2}, A\right]$ then
$\cos(A-B)-\cos B-\cos A \geq \cos\left(A-\frac{\pi-A}{2}\right)-\cos\frac{\pi-A}{2}-\cos A =$
$\sin\frac{3A}{2}-\sin\frac{A}{2}-\cos A = 2\cos A \cdot \sin\frac{A}{2}-\cos A = \cos A\left(2\sin\frac{A}{2}-1\right) \geq 0$
because $\cos A \geq 0$ and $2\sin\frac{A}{2} \geq 2\sin\frac{\pi}{4}-1 = \sqrt{2}-1 > 0$.

Since $\cos A\left(2\sin\frac{A}{2}-1\right) = 0 \Leftrightarrow A = \pi/2$ or $A = \pi/3$ and $B = \frac{\pi-A}{2}$
then equality in inequality $R+r \leq h_c$ holds iff $(A,B,C) = (\pi/2, \pi/4, \pi/4)$ or
$(A,B,C) = (\pi/3, \pi/3, \pi/3)$.